Optimal Design of Structures with Constraints on Natural Frequency

B. R. McCart*
Augustana College, Rock Island, Ill.

AND

E. J. Haug† and T. D. Streeter‡
Headquarters, U. S. Army Weapons Command, Rock Island, Ill.

A new method of solving infinite dimensional optimal design problems is presented in this paper. The class of problems considered includes minimum weight design of structures with constraints on strength and natural frequency. A steepest-descent boundary-value method is developed in detail for a three-member frame design problem and references are given for a more general development. A computational algorithm is presented which implements this steepest-descent method. Convergence for this algorithm is better than for algorithms that reduce the boundary-value problem to an initial-value problem by defining auxiliary variables and constraints. Results are presented which indicate the gains achieved by using optimal, nonuniform elements in a structure rather than uniform elements.

Nomenclature

D = differential operator
dP = step size in computational algorithm
E = Young's modulus of elasticity
I_0 = minimum allowable moment of inertia at any point along the frame
$I_i(s)$ = actual moment of inertia at any point s
$I_{JJ},I_{\psi J},I_{\psi\psi}$ = definite integrals
J = cost function
s = independent variable along frame
s_i = corner points of frame
T = matrix transpose
$u_i(s)$ = cross-sectional area of member i at point s
W = weighting matrix
y = state variable (vector)
z = state variable obtained from dependent variable transformation (vector)
α = smallest second moment of area of a cross-sectional with the given geometry having area one
β = multiplier for terminal constraint function
γ = superposition constant
γ^* = superposition constant
γ^+ = superposition constant
δ = variational operator
θ = Lagrange multiplier
Λ^Q = influence function
λ = vector of adjoint variables
ν = Lagrange multiplier
ρ = mass density of material
ϕ_i = inequality constraint function
ψ = terminal constraint function
ω = frequency of vibration

Introduction

ENGINEERS have been designing structures for centuries with the objective of developing a structure for a particular application that performs a specified function under given load conditions and is best in some sense. In recent years, aircraft and aerospace requirements have placed emphasis on design of structures for minimum weight. For a review of the progress of optimal structural design through 1967 see Ref. 1.

The development of computational techniques using high-speed digital computers has had a tremendous impact on many fields. Structural analysis, particularly, has benefited from these techniques. Efficient computational procedures, however, have not been developed for optimal structural design. Sheu and Prager[1] point out that because of the lack of computational methods, only optimal design problems of academic interest have been solved in the past.

This paper presents a direct computational method that has proved effective in the design of structural elements.[2] The method is quite general and is here applied to a three-member frame design problem. Indications are that, with refinement, the method may be used to solve large-scale, real-world problems.

The method developed here is an iterative process. A nonoptimal structure is first analyzed and as a result of the analysis, an improved structure is obtained. The process is repeated until no significant gains can be made. A pertinent feature of the method presented here is that it is implemented by an explicit computational algorithm. This method is very similar to steepest-ascent methods developed by Kelley, Bryson, et al.[3–5] in connection with aircraft guidance problems. The method differs from the classical steepest-ascent methods cited previously in that it makes strong use of the boundary value as opposed to initial-value nature of the problem. The general theory of the method is given in Ref. 6. Here, a computational algorithm will be derived only for the problem being treated. The specific problem treated in this paper is the design of a portal frame for minimum weight with restrictions on natural frequency and strength. A method of steepest-descent is developed in detail for this problem, using the boundary-value nature of the equations of vibration. The derivation presented here illustrates all the theory required for more general problems.

A collection of numerical cases for the portal frame problem are run and the results reported. The percent saving of material achieved by utilizing the optimum frame rather than a frame with uniform members is significant. Limited experience indicates that the more complex the structure,

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* Professor of Physics, Physics Department.
† Mechanical Engineer, Systems Analysis Directorate.
‡ Mathematician, Systems Analysis Directorate.

the greater the weight saving by using the optimal structure rather than a structure composed of uniform elements.

The authors feel that the principal contribution of this paper to the optimal design literature is the constructive method presented. Steepest-descent methods utilizing the boundary-value nature of the structural analysis problem appear to be very effective. In particular, if the designer is capable of analyzing a structure, then he has the tools required to begin optimal design.

Formulation of the Problem

The design problem treated here may be stated as follows: find the distribution of material along the members of the portal frame in Fig. 1 such that the second moment of the cross-sectional area at each point is no less than a given constant, the natural frequency of the frame is no less than a given level, and the frame is as light in weight as possible. If the second moment of the cross-sectional area $u_i(s)$ is $I_i(s)$, $i = 1,2,3$, and the lowest allowed natural frequency is ω, then this problem may be stated mathematically as

$$\text{minimize } J = \int_0^{s_1} u_1(s)ds + \int_{s_1}^{s_2} u_2(s)ds + \int_{s_2}^{s_3} u_3(s)ds \quad (1)$$

subject to the conditions

$$\phi_i(s,u) = I_0 - I_i(s) \leq 0, \; i = 1,2,3, \; 0 \leq s \leq s_3 \quad (2)$$

and

$$\text{natural frequency} \geq \omega \quad (3)$$

where J is volume of the material in the frame. The material is taken to have constant density, so minimum weight is equivalent to minimum volume.

The constraint Eq. (2) is interpreted as a strength constraint. That is, the frame is to be subjected to loads that require that Eq. (2) hold. This constraint could be stated more realistically in terms of given loads and the associated stresses without theoretical difficulties. The algebra, however, would be messy. For a discussion of other constraints see Ref. 2.

The constraint Eq. (3) is imposed so as to prevent resonance problems due to excitation of the structure. For example, an automatic weapon attached to a helicopter frame will cause large amplitude oscillation to occur if the weapon fires at a rate that excites the airframe in its fundamental mode. Analogous problems occur with instruments, engines, etc. that are attached to flexible structures.

In order to complete the statement of the problem something must be said about how $I_i(s)$ is related to $u_i(s)$. In the problem treated here, the basic geometry of the cross sections is chosen. It could be square, circular, annular, wide flange, angle, etc. This basic shape is maintained but all dimensions of the cross section are allowed to vary along the length of the members with the same ratio. The ratio varies with s and hence determines the tapering of the members. It is shown in Ref. 2, p. 57 that in this case

$$I_i(s) = \alpha u_i^2(s), \; i = 1,2,3 \quad (4)$$

The factor α is the smallest second moment of area of a cross section with the given geometry having area one.

The frequency constraint Eq. (3) is treated by a two step procedure. If the natural frequency of a frame with $I_i(s) = I_0$, $i = 1,2,3$ is greater than ω, then this is the optimal frame since an addition of material anywhere will increase $I(s)$ at that point, which will at least not decrease the natural frequency of the frame. If on the other hand, the natural frequency of the frame with $I_i(s) = I_0$, $i = 1,2,3$ is less than ω, it will be required that the natural frequency of the optimal frame equal ω. This is the only case that requires analysis so it will be considered through the remainder of the paper.

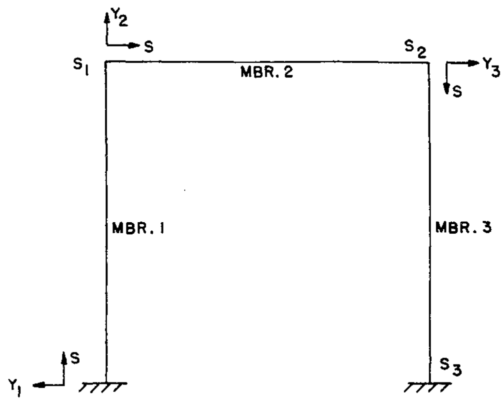

Fig. 1 Portal frame.

In order to relate natural frequency to the design variables $u_i(s)$, the vibratory motion of the frame must be treated. The displacement functions y_i are, in the case where resonance occurs

$$y_i(s,t) = y_i(s) \sin\omega t, \; i = 1,2,3 \quad (5)$$

Here, $y_i(s)$ is defined only on the interval (s_{i-1},s_i). The differential equations for the mode shapes are then

$$(d^2/ds^2)\{E\alpha u_i^2(s)[d^2y_i(s)/ds^2]\} = \rho\omega^2 u_i(s)y_i(s), \; i = 1,2,3 \quad (6)$$

where E is Young's modulus and ρ is the density of the material from which the frame is constructed. It will be observed that Eq. (6) is the general form of the elastic beam equation.

The boundary conditions on $y_i(s)$ are

deflections:

$$y_1(0) = y_2(s_1) = y_2(s_2) = y_3(s_3) = 0 \quad (7a)$$

$$y_1(s_1) = -y_3(s_2) \quad (7b)$$

slopes:

$$y_1'(0) = y_3'(s_3) = 0 \quad (7c)$$

$$y_1'(s_1) = y_2'(s_1) \quad (7d)$$

$$y_2'(s_2) = y_3'(s_2) \quad (7e)$$

moments:

$$E\alpha u_1^2(s_1)y_1''(s_1) = E\alpha u_2^2(s_1)y_2''(s_1) \quad (7f)$$

$$E\alpha u_2^2(s_2)y_2''(s_2) = E\alpha u_3^2({}_3s_2)y_3''(s_2) \quad (7g)$$

shear

$$-\omega^2\rho\left[\int_{s_1}^{s_2} u_2(s)ds\right] y_1(s_1) = \frac{d}{ds}\,[E\alpha u_1^2(s_1)y_1''(s_1)] + \frac{d}{ds}\,[E\alpha u_3^2(s_2)y_3''(s_2)] \quad (7h)$$

where primes denote (d/ds). Equation (7h) is obtained from Newton's equation of motion of the horizontal member.

The boundary conditions (7) and differential equations (6) are homogeneous, so the solution is specified only to within an arbitrary constant. For mathematical convenience in obtaining a unique eigenfunction, one additional nonhomogeneous boundary condition may be added. In the present problem, let

$$E\alpha u_1^2 y_1''(0) = \text{const} \quad (8)$$

where the constant can be whatever magnitude the designer desires. The value of the constant determines the magnitude of oscillations of the frame. Note that Eqs. (7) and (8) together comprise thirteen boundary conditions, so for just any $u_i(s)$, $i = 1,2,3$, one should not expect a solution of the system Eqs. (6–8) to exist.

For convenience in later development, it will be advantageous to put Eqs. (6) in first-order form. This is done by defining new variables $z_i(s)$, $i = 1, \ldots, 12$ as

$$z_1 = y_1 \tag{9a}$$

$$z_2 = y_1' \tag{9b}$$

$$z_3 = E\alpha u_1^2 y_1'' \tag{9c}$$

$$z_4 = -(E\alpha u_1^2 y_1'')' \tag{9d}$$

$$z_5 = y_2 \tag{9e}$$

$$z_6 = y_2' \tag{9f}$$

$$z_7 = E\alpha u_2^2 y_2'' \tag{9g}$$

$$z_8 = -(E\alpha u_2^2 y_2'')' \tag{9h}$$

$$z_9 = y_3 \tag{9i}$$

$$z_{10} = y_3' \tag{9j}$$

$$z_{11} = E\alpha u_3^2 y_3'' \tag{9k}$$

$$z_{12} = -(E\alpha u_3^2 y_3'')' \tag{9l}$$

In a more compact notation, $z = [z_1, \ldots, z_{12}]^T$ and $u = [u_1,u_2,u_3]^T$ are column vectors called the state variable and the design variable, respectively. With these definitions, the differential equations (6) are replaced by the first-order system

$$\left.\begin{aligned} -z_4' &= \rho\omega^2 u_1 z_1 \\ -z_3' &= z_4 \\ z_2' &= [(1/E\alpha u_1^2)]z_3 \\ z_1' &= z_2 \end{aligned}\right\} 0 \leq s \leq s_1 \tag{10a}$$

$z_i(s)$, $i = 1,2,3,4$ are only defined in $s_0 \leq s \leq s_1$

$$\left.\begin{aligned} -z_8' &= \rho\omega^2 u_2 z_5 \\ -z_7' &= z_8 \\ z_6' &= [(1/E\alpha u_2^2)]z_7 \\ z_5' &= z_6 \end{aligned}\right\} s_1 \leq s \leq s_2 \tag{10b}$$

$z_i(s)$, $i = 5,6,7,8$ are only defined in $s_1 \leq s \leq s_2$

$$\left.\begin{aligned} -z_{12}' &= \rho\omega^2 u_3 z_9 \\ -z_{11}' &= z_{12} \\ z_{10}' &= [(1/E\alpha u_3^2)]z_{11} \\ z_9' &= z_{10} \end{aligned}\right\} s_2 \leq s \leq s_3 \tag{10c}$$

$z_i(s)$, $i = 9,10,11,12$ are only defined in $s_2 \leq s \leq s_3$.

The arrangement of equations in Eqs. (10) is chosen so that the system of equations is self-adjoint.[7]

For purposes of manipulation, the set of twelve differential equations (10) may be put into a more compact notation. Namely

$$Dz = 0 \tag{11}$$

where D is the differential operator

$$D = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}_{12\times 12}$$

and the D_i are the operators

$$D_i = \begin{bmatrix} -\rho\omega^2 u_i & 0 & 0 & -d/ds \\ 0 & 0 & -d/ds & -1 \\ 0 & d/ds & -(1/E\alpha u_i^2) & 0 \\ d/ds & -1 & 0 & 0 \end{bmatrix}_{4\times 4} \tag{12}$$

Note that the operator D depends on the design variable u.

The boundary conditions, Eqs. (7) and (8), in the notation of Eq. (9), are

$$z_1(0) = z_5(s_1) = z_5(s_2) = z_9(s_3) = 0,\ z_2(0) = 0 \tag{13a}$$

$$z_1(s_1) = -z_9(s_2),\ z_2(s_1) = z_6(s_1) \tag{13b}$$

$$z_6(s_2) = z_{10}(s_2),\ z_3(0) = \text{const} \tag{13c}$$

$$z_3(s_1) = z_7(s_1),\ z_7(s_2) = z_{11}(s_2) \tag{13d}$$

$$\omega^2\rho \left[\int_{s_1}^{s_2} u_2(s)ds\right] z_1(s_1) = z_4(s_1) + z_{12}(s_2) \tag{13e}$$

and

$$\psi = z_{10}(s_3) = 0 \tag{14}$$

The boundary conditions are separated into Eqs. (13) and (14) for computational reasons. It is possible, for a given choice of design variable $u_i(s)$, $i = 1,2,3$ to integrate the homogeneous differential equations (10) with boundary conditions Eq. (13) to obtain a nonzero solution. A superposition scheme described in Appendix A is used for this purpose since the problem is linear.[8] The condition Eq. (14) will probably not be satisfied, so adjustments which are to be made in the $u_i(s)$ will be required to reduce the error in satisfying Eq. (14).

Method of Steepest Descent

The method used to solve this optimal design problem will be iterative in nature. An engineering estimate will first be made of the best distribution of material in the structure, say $u^0(s)$. The associated state variable, determined by Eqs. (11) and (13), is denoted $z^0(s)$. A small change $\delta u(s)$ in $u^0(s)$ will then be determined which reduces J and satisfies the constraints of the problem. The design variable $u^1(s) = u^0(s) + \delta u(s)$, will then be taken as an improved estimate of the optimal structure and the process will be repeated. The purpose of this section is to develop a method for determining $\delta u(s)$.

If $\delta u(s)$ is small, then the functions involved in the optimal design problem may be approximated by first-order Taylor expansions. The small change $\delta u(s)$ in $u^0(s)$ also causes a small change $\delta z(s)$ in $z^0(s)$. This change in the state variable must satisfy the equation

$$D\delta z + (\partial/\partial u)(Dz^0)\delta u = 0 \tag{15}$$

where the operator D is defined in Eq. (12) and the matrix calculus notation

$$\partial Q/\partial w = [\partial Q_i/\partial w_j]kxl \tag{16}$$

is used where $Q = [Q_1(w), \ldots, Q_k(w)]^T$ is a vector function of the vector variable $w = [w_1, \ldots, w_l]^T$.

To illustrate the use of this notation, the matrix $(\partial/\partial u)(Dz^0)$ is

$$\frac{\partial}{\partial u}(Dz^0) = \begin{bmatrix} -\rho\omega^2 z_1^0 & 0 & 0 \\ 0 & 0 & 0 \\ 2z_3^0/E\alpha u_1^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\rho\omega^2 z_5^0 & 0 \\ 0 & 0 & 0 \\ 0 & 2z_7^0/E\alpha u_2^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho\omega^2 z_9^0 \\ 0 & 0 & 0 \\ 0 & 0 & 2z_{11}^0/E\alpha u_3^3 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Since $z^0(s)$ is required to satisfy Eq. (13), $\delta z(s)$, which is the change in $z^0(s)$ caused by $\delta u(s)$, must satisfy

$$\delta z_1(0) = \delta z_2(0) = \delta z_3(0) = \delta z_5(s_1) = \delta z_5(s_2) = \delta z_9(s_3) = 0 \quad (18a)$$

$$\delta z_1(s_1) = -\delta z_9(s_2) \quad (18b)$$

$$\delta z_2(s_1) = \delta z_6(s_1) \quad (18c)$$

$$\delta z_6(s_2) = \delta z_{10}(s_2) \quad (18d)$$

$$\delta z_3(s_1) = \delta z_7(s_1) \quad (18e)$$

$$\delta z_7(s_2) = \delta z_{11}(s_2) \quad (18f)$$

$$\omega^2\rho\left[\int_{s_1}^{s_2}\delta u_2(s)ds\right]z_1^0(s_1) + \omega^2\rho\left[\int_{s_1}^{s_2}u_2(s)ds\right]\delta z_1(s_1) = \delta z_4(s_1) + \delta z_{12}(s_2) \quad (18g)$$

Changes in the functions of interest in this problem are approximated by

$$\delta J = \int_0^{s_1}\delta u_1(s)ds + \int_{s_1}^{s_2}\delta u_2(s)ds + \int_{s_2}^{s_3}\delta u_3(s)ds \quad (19)$$

$$\delta\phi_i = -2\alpha_i^0(s)\delta u_i(s),\ i = 1,2,3 \quad (20)$$

and

$$\delta\psi = \delta z_{10}^0(s_3) \quad (21)$$

The iterative algorithm that is to be used to solve the optimal design problem can now be outlined in greater detail. The primary objective in choosing a $\delta u(s)$ is to reduce J as much as possible. That is, minimize (maximize the negative of) δJ. The constraints, Eqs. (2) and (14), however, must be considered in choosing $\delta u(s)$. Constraint Eq. (2) will be treated initially by requiring that ϕ_i remain zero at each point where it is zero before the iteration begins.

The constraint Eq. (14) will be treated by first checking ψ after the boundary-value problem, Eqs. (10–13), has been solved with $u(s) = u^0(s)$. If $\psi = z_{10}^0(s_3) \neq 0$, then a change in ψ that will reduce the error will be required. This change is

$$\delta\psi = -\beta z_{10}^0(s_3) \quad (22)$$

where β, $0 < \beta \leq 1$, is chosen by the optimizer. The reason for making β small, particularly at first when ψ is large, is that the predicted change in ψ is based on a first-order approximation of $\delta\psi$. If large changes are required, this approximation may be poor.

The problem of determining $\delta u(s)$ to minimize δJ of Eq. (19) subject to Eqs. (20) and (22) is now almost a standard problem of the calculus of variations. The difficulty, of course, is that Eq. (22) is not given explicitly in terms of $\delta u(s)$. It is clear that $\delta z_{10}(s_3)$ depends on $\delta u(s)$ through the boundary-value problem Eqs. (15–18). The dependence of $\delta z_{10}(s_3)$ on $\delta u(s)$ can be determined explicitly using a basic adjoint relationship from differential equations (Ref. 7, p. 182). This relationship may be verified by performing integration by parts. It is

$$\int_0^{s_3}\{\lambda^T D\delta z - \delta z^T D\lambda\}ds = (-\lambda_1\delta z_4 - \lambda_2\delta z_3 + \lambda_3\delta z_2 + \lambda_4\delta z_1)\Big|_0^{s_1} + (-\lambda_5\delta z_8 - \lambda_6\delta z_7 + \lambda_7\delta z_6 + \lambda_8\delta z_5)\Big|_{s_1}^{s_2} + (-\lambda_9\delta z_{12} - \lambda_{10}\delta z_{11} + \lambda_{11}\delta z_{10} + \lambda_{12}\delta z_9)\Big|_{s_2}^{s_3} \quad (23)$$

This is an identity that holds for continuously differentiable vector function $\lambda(s) = [\lambda_1(s), \ldots, \lambda_{12}(s)]^T$.

Eliminating all known and related $\delta z_i(s_j)$ from Eq. (23) by using Eq. (18)

$$\int_0^{s_3}\{\lambda^T D\delta z - \delta z^T D\lambda\}ds + \lambda_1(s_1)z_1(s_1)\omega^2\rho\int_{s_1}^{s_2}\delta u_2(s)ds = \lambda_1(0)\delta z_4(0) + [\lambda_3(s_1) - \lambda_7(s_1)]\delta z_6(s_1) + [\lambda_6(s_1) - \lambda_2(s_1)]\delta z_7(s_1) + \lambda_5(s_1)\delta z_8(s_1) - \lambda_5(s_2)\delta z_8(s_2) - [\lambda_4(s_1) + \lambda_{12}(s_2) - \omega^2\rho\lambda_1(s_1)\int_{s_1}^{s_2}u_2(s)ds]\delta z_9(s_2) + [\lambda_7(s_2) - \lambda_{11}(s_2)]\delta z_{10}(s_2) + \lambda_{11}(s_3)\delta z_{10}(s_3) + [\lambda_{10}(s_2) - \lambda_6(s_2)]\delta z_{11}(s_2) - \lambda_{10}(s_3)\delta z_{11}(s_3) + [\lambda_9(s_2) + \lambda_1(s_1)]\delta z_{12}(s_2) - \lambda_9(s_3)\delta z_{12}(s_3) \quad (24)$$

If $\lambda_{11}(s_3)$ is chosen as one and the remaining coefficients of the $\delta z_i(s_j)$ appearing in Eq. (24) are chosen to be zero, then the right side of Eq. (24) is $\delta\psi$. In order to eliminate dependence of the left side of Eq. (24) on $\delta z(s)$, choose that $\lambda(s)$ that satisfies

$$D\lambda = 0,\ 0 \leq s \leq s_3 \quad (25)$$

where $\lambda_i(s)$, $i = 1,2,3,4$ are only defined for $s_0 \leq s \leq s_1$, $\lambda_i(s)$, $i = 5,6,7,8$ are only defined for $s_1 \leq s \leq s_2$, and $\lambda_i(s)$, $i = 9,10,11,12$ are only defined in $s_2 \leq s \leq s_3$. Using this differential equation for $\lambda(s)$ and the boundary conditions described above, a solution for $\lambda(s)$ may be obtained. These boundary conditions are

$$\lambda_1(0) = 0 \quad (26a)$$

$$\lambda_3(s_1) - \lambda_7(s_1) = 0 \quad (26b)$$

$$\lambda_6(s_1) - \lambda_2(s_1) = 0 \quad (26c)$$

$$\lambda_5(s_1) = 0 \quad (26d)$$

$$\lambda_5(s_2) = 0 \quad (26e)$$

$$\lambda_4(s_1) + \lambda_{12}(s_2) - \omega^2\rho\lambda_1(s_1)\int_{s_1}^{s_2}u_2(s)ds = 0 \quad (26f)$$

$$\lambda_7(s_2) - \lambda_{11}(s_2) = 0 \quad (26g)$$

$$\lambda_{11}(s_3) = 1 \quad (26h)$$

$$\lambda_{10}(s_2) - \lambda_6(s_2) = 0 \quad (26i)$$

$$\lambda_{10}(s_3) = 0 \quad (26j)$$

$$\lambda_9(s_2) + \lambda_1(s_1) = 0 \quad (26k)$$

$$\lambda_9(s_3) = 0 \quad (26l)$$

Substituting from Eqs. (25) and (26) into Eq. (24)

$$\int_0^{s_3}\lambda^T D\delta z ds + \lambda_1(s_1)z_1(s_1)\omega^2\rho\int_{s_1}^{s_2}\delta u_2(s)ds = \delta z_{10}(s_3)$$

Substituting from Eqs. (15) and (21) into this equation in order to eliminate dependence on $\delta z(s)$

$$\delta\psi = \int_0^{s_3} l^{\psi T}(s)\delta u(s)ds \quad (27)$$

where

$$l^{\psi}(s)_{3\times 1} = \begin{cases} -\left[\lambda^T\dfrac{\partial}{\partial u}(Dz^0)\right]^T \text{ for } 0 \leq s \leq s_1 \text{ and } s_2 \leq s \leq s_3 \\ -\left[\lambda^T\dfrac{\partial}{\partial u}(Dz^0)\right]^T + \lambda_1(s_1)z_1(s_1)\omega^2\rho\begin{bmatrix}0\\1\\0\end{bmatrix}, \text{ for } s_1 \leq s \leq s_2 \end{cases} \quad (28)$$

Note now that $\delta\psi$ is given explicitly in terms of δu once $\lambda(s)$ and $z^0(s)$ have been determined. The problem of determining δu is now of the form of a problem of the calculus of variations. Before stating this problem, it should be noted that $\delta u(s)$ was required to be small throughout all the foregoing analysis. In order to insure that this is the case, it is required that

$$dP^2 = \int_0^{s_3} \delta u^T(s) W(s) \delta u(s) ds \tag{29}$$

where dP is small and $W(s)$ is a weighting matrix that may be chosen to adjust the importance associated with each of the components of $\delta u(s)$. Both of these quantities are to be chosen by the optimizer.

The nature of the inequality constraints of Eq. (2) may be used to advantage here. If some $\phi_i(s,u^0) = I_0 - \alpha(u^0(s))^2 = 0$ at some point, then in the next iteration $\delta\phi_i = 0$ is required. But, since $\phi_i(s,u^0)$ depends only on the component $u_i^0(s)$ of $u^0(s)$, then $\delta\phi_i(s,u^0) = 0$ is equivalent to $\delta u_i(s) = 0$. This requirement may be reflected in the linearized constraints of the problem directly. Using this idea, the simplified problem is to find $\delta u(s)$ that minimizes

$$\delta J = \int_0^{s_3} \Lambda^{J^T}(s) \delta u(s) ds \tag{30}$$

subject to

$$\int_0^{s_3} \Lambda^{\psi^T}(s) \delta u(s) ds = -\beta z_{10}^0(s_3) \tag{31}$$

and Eq. (29), where

$$\underset{3\times 1}{\Lambda^J(s)} = \begin{bmatrix} H(\phi_1) & 0 & 0 \\ 0 & H(\phi_2) & 0 \\ 0 & 0 & H(\phi_3) \end{bmatrix} l^J(s) \tag{32}$$

$$\underset{3\times 1}{l^{J^T}(s)} = \begin{cases} [1,0,0], \text{ in } 0 \le s \le s_1 \\ [0,1,0], \text{ in } s_1 \le s \le s_2 \\ [0,0,1]. \text{ in } s_2 \le s \le s_3 \end{cases}$$

$$\underset{3\times 1}{\Lambda^\psi(s)} = \begin{bmatrix} H(\phi_1) & 0 & 0 \\ 0 & H(\phi_2) & 0 \\ 0 & 0 & H(\phi_3) \end{bmatrix} l^\psi(s) \tag{33}$$

and

$$H(\phi_i) = \begin{cases} 1, \ \phi_i < 0 \\ 0, \text{ otherwise} \end{cases}$$

In order to find $\delta u(s)$, adjoin the constraints to δJ using Lagrange multipliers to obtain

$$\delta \bar{J} = \int_0^{s_3} [\Lambda^{J^T}(s)\delta u(s) + \nu\Lambda^{\psi^T}(s)\delta u(s) + \theta \delta u^T(s) W(s) \delta u(s)] ds$$

where ν and θ are undetermined multipliers. As shown in any calculus of variations text it is necessary that $\delta^2\bar{J} = \delta(\delta\bar{J}) = 0$ for all $\delta^2 u(s)$. That is

$$\delta^2\bar{J} = \int_0^{s_3} [\Lambda^{J^T}(s) + \nu\Lambda^{\psi^T}(s) + 2\theta\delta u^T(s) W(s)] \delta^2 u(s) ds = 0$$

Therefore, it is necessary that the integrand be zero for all s, so

$$\delta u(s) = -(1/2\theta) W^{-1}(s)[\Lambda^J(s) + \nu\Lambda^\psi(s)] \tag{34}$$

The multipliers ν and θ are found by substituting $\delta u(s)$ from Eq. (34) into Eqs. (31) and (29), respectively. The solution is

$$\nu = [2\theta\beta z_{10}^0(s_3) - I_{\psi J}]/I_{\psi\psi}$$

$$2\theta = \{[I_{JJ} - I_{\psi J}^2]/I_{\psi\psi}/[dP^2 - [\beta z_{10}^0(s_3)]^2/I_{\psi\psi}]\}^{1/2}$$

and

$$\delta u(s) = -\left\{\frac{dP^2 - [\beta z_{10}^0(s_3)]^2/I_{\psi\psi}}{I_{JJ} - I_{\psi J}^2/I_{\psi\psi}}\right\}^{1/2} \cdot W^{-1}(s)\left[\Lambda^J(s) - \frac{I_{\psi J}}{I_{\psi\psi}}\Lambda^\psi(s)\right] - \frac{\beta z_{10}^0(s_3)}{I_{\psi\psi}} W^{-1}(s)\Lambda^\psi(s) \tag{35}$$

where

$$I_{JJ} = \int_0^{s_3} \Lambda^{J^T}(s) W^{-1}(s) \Lambda^J(s) ds \tag{36}$$

$$I_{\psi J} = \int_0^{s_3} \Lambda^{\psi^T}(s) W^{-1}(s) \Lambda^J(s) ds \tag{37}$$

$$I_{\psi\psi} = \int_0^{s_3} \Lambda^{\psi^T}(s) W^{-1}(s) \Lambda^\psi(s) ds \tag{38}$$

Note that the numerator of the fraction under the one half power in Eq. (35) could be negative if $\beta z_{10}^0(s_3)$ is too large. This possibility occurs because the requested improvement, $\beta z_{10}^0(s_3)$, in this error, requires a larger dP. When this occurs, the usual remedy is to simply reduce β which essentially requests less improvement in the error.

The iterative procedure for solving the optimal design problem is to estimate the optimum design as $u^0(s)$. A correction $\delta u(s)$ is then computed according to Eq. (35) and $u^1(s) = u^0(s) + \delta u(s)$ is then an improved estimate of the optimum design. The new estimate then replaces $u^0(s)$ and the process is repeated to obtain still another improvement.

For the first few iterations, the primary effort of the process is devoted to reducing the terminal error toward zero. Once this error is near zero, more effort is directed toward decreasing J. If, in order to ignore constraint error corrections, one puts $\beta = 0$ in Eq. (35) then substitution of the resulting expression for $\delta u(s)$ into Eq. (30) yields

$$\delta J = -dP[I_{JJ} - I_{\psi J}^2/I_{\psi\psi}]^{1/2}$$

Therefore, the quantity $[I_{JJ} - I_{\psi J}^2/I_{\psi\psi}]^{1/2}$ may be thought of as the rate of change of J with respect to a fictitious variable P. At the optimum, this rate of change, or gradient should be zero. This gradient is a convenient parameter to monitor as a check on convergence. For more discussion on interpretation of quantities appearing in this development, see Refs. 2 and 4–6.

The foregoing discussion may now be summarized in the form of a computational algorithm.

Computational Algorithm

Step 1. Make an engineering estimate $u^0(s)$ of the optimum design that satisfies Eq. (2).

Table 1 Optimal u_1 and u_3 for aluminum

s^ω	Optimal design variable $u(s)$ at different stations along the first member for three frequencies of vibration 2000	3000	4000
0.0	1.202	2.133	3.173
0.4	1.159	2.054	3.057
0.8	1.115	1.974	2.939
1.2	1.069	1.892	2.819
1.6	1.022	1.809	2.696
2.0	0.974	1.723	2.570
2.4	0.924	1.635	2.440
2.8	0.873	1.545	2.308
3.2	0.819	1.452	2.171
3.6	0.763	1.357	2.031
4.0	0.705	1.259	1.887
4.4	0.645	1.158	1.740
4.8	0.581	1.055	1.590
5.2	0.514	0.950	1.438
5.6	0.444	0.843	1.284
6.0	0.369	0.734	1.130
6.4	0.350	0.625	0.977
6.8	0.350	0.514	0.826
7.2	0.350	0.401	0.677
7.6	0.350	0.350	0.350
8.0	0.350	0.350	0.384
8.4	0.350	0.350	0.350
8.8	0.350	0.350	0.350
9.2	0.350	0.350	0.350
9.6	0.350	0.350	0.350
10.0	0.350	0.350	0.350

Step 2. Integrate Eqs. (10) with boundary conditions Eqs. (13) using the method of the Appendix.

Step 3. Integrate Eqs. (25) with boundary conditions Eqs. (26) using the method of the Appendix.

Step 4. Evaluate $\Lambda^J(s)$ and $\Lambda^\psi(s)$ in Eqs. (32) and (33), respectively. If $\phi_i(s) = 0$ at some grid points and is strictly negative at either of the adjacent grid points, then $\phi_i(s)$ is treated as if it were strictly negative in Eqs. (32) and (33).

Step 5. Evaluate the integrals for I_{JJ}, $I_{\psi J}$, and $I_{\psi\psi}$ in Eqs. (36–38), respectively.

Step 6. Examine the error $z_{10}(s_3)$, and choose β and dP.

Step 7. Compute $dP^2 - [\beta z_{10}(s_3)]^2/I_{\psi\psi}$. If this quantity is negative, replace β by

$$\xi = [dP^2 I_{\psi\psi}/z_{10}^2(s_3)]^{1/2}$$

Step 8. Evaluate $\delta u(s)$ in Eqs. (35) and compute the new estimate

$$u^1(s) = u^0(s) + \delta u(s)$$

Step 9. Every few (10 to 20) iterations, completely neglect the constraints, Eqs. (2) (i.e., assume all $\phi_i < 0$) and compute a $\delta u(s)$ with dP very small.

Step 10. Check the constraints, Eqs. (2), using $u^1(s)$. If $\phi_i(s) > 0$ for any i and s, adjust $u_i^1(s)$ so that $\phi_i(s) = 0$.

Step 11. If $I_{JJ} - I_{\psi J}^2/I_{\psi\psi}$ is sufficiently near zero and no appreciable change in $u(s)$ is obtained, the iterative process is terminated. Otherwise, let $u^1(s)$ play the role of $u^0(s)$ and return to step 2.

This iterative process has given good results when used by an optimizer with only a moderate amount of experience. Some experience is helpful in the choice of dP and β. In the problem treated here, $\beta = 1$ was used and dP was reduced as the gradient of the process $I_{JJ} - I_{\psi J}^2/I_{\psi\psi}$ converged toward zero. A more detailed discussion is given in the following section.

The qualification in step 4 of the iterative process is inserted to counteract a difficulty introduced by the method of handling inequality constraints, Eqs. (2). Without this qualification, if $\phi_i(s,u) = 0$ in any iteration, then $\phi_i(s,u) = 0$ for all successive iterations. It may happen that early in the iterative process, while the terminal error is being reduced, $\phi_i(s,u) = 0$ may occur. After the terminal error is driven toward zero, however, the design variable should theoretically be adjusted so that $\phi_i(s,u) < 0$ may occur. Without the qualification in step 4, this is impossible.

Results

The geometry of the cross section used for this study was circular. However, any cross section could have been used simply by changing the constant α.

Typical materials that might be used for aircraft structures are aluminum and steel. Results for each of these materials are given in Tables 1 and 2. The design variable $u(s)$ shows the mass distribution characteristic for the minimum weight portal frame whose frequency of vibration must be equal to or greater than a specified value along with satisfying certain strength requirements. It can be seen from Fig. 2 that a significant material saving is possible in comparison to the portal frame with members of constant cross section.

Table 2 Comparison of optimal frames

Frequency, rad/sec	Weight of uniform frame Al	Steel	Weight of optimal frame Al	Steel	Percent weight reduction Al	Steel
2000	3.748	10.750	1.639	4.434	56.27%	58.75%
3000	8.434	24.187	2.504	6.828	70.31%	71.77%
4000	14.994	[a]	3.487	[a]	76.74%	[a]

[a] See comment in Conclusions Sec.

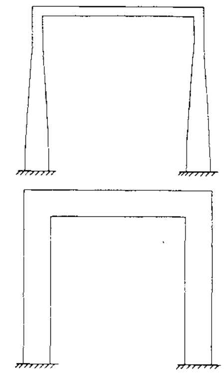

Fig. 2 Profiles of optimal and uniform frames for Aluminum at 3000 rad/sec.

The inequality $I(s) \geq I_0$ was imposed as a strength constraint. It should be noted that the inequality strength constraints prevents a zero cross section at any point along the frame. To make a comparison of results for the various materials realistic, I_0 was chosen for each material so that for a given bending moment, the yield stress for each material was attained if the cross section had the corresponding second moment of inertia I_0 for that material. The values of I_0 are given, along with material properties in Table 3.

Because of the high-speed digital computer, an automatic method had to be devised for selecting the step size dP and the change in the terminal constraint $d\psi$ at each iteration. $d\psi$ was set equal to $-z_{10}(s_3)$. The step size dP is made up of two parts. That is

$$dP = dP_1 + dP_2$$

where dP_1 will contribute the change that is necessary to drive the terminal constraint function to zero. The second part dP_2 will contribute an additional quantity which drives the objective function to a minimum

$$dP_1 = d\psi^T I_{\psi\psi}^{-1} d\psi$$

$$dP_2 = (I_{JJ} - I_{\psi J}^T I_{\psi\psi}^{-1} I_{\psi J})\Delta J$$

The first term in dP_2 is the gradient of the process squared. The quantity ΔJ is the desired change at each iteration in the cost function. The terminal constraint and the gradient will approach zero as the solution approaches optimality. Thus, dP_1 and dP_2 will approach zero. At each iteration dP is not allowed to exceed a specified value, namely, one one-hundredth of the volume of the uniform structure that has the given natural frequency. If too large a dP is used, the solution will diverge. A request for a small dP keeps the solution stable and results in fairly fast convergence. However, if too small a dP is used, convergence is very slow.

An interesting side point is the speed of convergence. The lower frequencies required approximately 25 iterations while the higher frequencies were taking around 60 iterations. It must be remembered, however, that over twice as much material had to be taken away at the higher frequencies since the required uniform cross section was used as the estimated design variable. In order to reduce the number of iterations at the higher frequencies, one could use a larger step size or a better estimate of the design variable that could be obtained from the lower frequencies. The latter of the two is preferable.

In order to improve convergence properties at each iteration, design variables on the constraint boundary ($\phi_i = 0$ for some i) were monitored and only those points on the constraint boundary whose adjacent point was in the interior of the constraint set were allowed to come off the constraint boundary.

Another point of interest is the behavior of the terminal constraint function ψ, the objective function J, and the distribution of mass during the iterative process when u^0 is

Table 3 Material properties

Property	Steel	Aluminum
ρ lbs-sec^2/in.^{4}	7.4344×10^{-4}	2.6163×10^{-4}
σ lbs/in.^{2}	70,000	60,000
E lbs/in.^{2}	29×10^6	10.3×10^6
I_0 in.^{4}	0.008	0.009825
α dimensionless	0.08	0.08
u^0 in.^{2} (for 2000 rad/sec)	1.2485	1.2371
u^0 in.^{2} (for 3000 rad/sec)	2.8090	2.7835
u^0 in.^{2} (for 4000 rad/sec)	4.9940	4.9484

chosen as

$$u_1(t) = 0.8, u_2(t) = 0.5, u_3(t) = 0.8$$

so that the uniform members do not have the desired natural frequency when $\omega \geq 2000$ rad/sec for a minimum J. During the first few iterations the ψ constraint is driven to zero and remains there. The value of the gradient and J increase about five percent until $\psi = 0$. For the remaining iterations, J approaches its optimum value while the gradient progressively approaches zero. Also, during the first few iterations, the first and third members approach their optimum shape while the second member goes to the constraint boundary at the middle of the beam. Then, for the remaining iterations, the second member goes to the constraint boundary everywhere working outward from the middle of the beam while the first and third members achieve their optimal shape as shown in Fig. 2.

The initial conditions for the state and adjoint equations were given in Eqs. (13) and (26). The lengths of each of the frame members are ten inches and the values for the remaining parameters are given in Table 3.

Conclusions

The multi-member optimal design problem considered in this paper approaches the level of complexity encountered in real-world structural design problems. The formulation of this problem is direct in nature and indicates how more complicated problems may be approached. Except for the number of equations and unknowns involved, many member structural design problems may be formulated just as the three member problem considered here.

Although steepest-descent methods based on the work of Bryson, Denham, and others[4,5] can be used to solve optimal design problems, the direct boundary-value approach presented here appears to be better suited to structural design problems. The methods of Refs. 4 and 5 require that the governing boundary-value problem be converted to an initial value problem with additional variables and constraints. These artificial features tend to complicate the computational algorithm and slow convergence. Further, since in that formulation parameters must be estimated that have little physical significance, divergence can occur if poor initial estimates are made. The boundary-value method presented here has none of these shortcomings. For a general treatment of this method, the reader is referred to Ref. 6.

Since the steepest-descent algorithm converges to a frame with minimum volume, the algorithm will converge to a structure that vibrates at the required frequency but possibly at a higher mode if $\omega \geq 4187$ for aluminum and $\omega \geq 3959$ for steel. The fundamental frequency of this structure may be below the level required so this structure is, in reality, not admissible. For example, in the frame treated here for steel with $\omega \geq 4000$ the method converges to a frame with a symmetric mode of vibration (second mode) of the required frequency. This difficulty has been overcome by a more general method to be presented soon.[9]

The saving of material achieved using optimum designs rather than designs employing uniform members is significant, as pointed out in the preceding section. It has been the experience of the authors that the more complex the structure, the greater the improvement obtained in resorting to optimum, nonuniform members.

Finally, the constructive method presented here for solution of optimal structural design problems appears to be very promising. The authors feel that it represents a step forward in the important problem of finding practical ways of constructing optimal structural designs.[1] This method may be used in a direct manner for optimal design of a system if analysis of the behavior of that system can be efficiently performed.

Results for the problem solved here were presented in Ref. 10. A computational error appeared in those calculations, unfortunately, so the optimum frame presented here should be regarded as a correction to those results.

Appendix: Integration of State and Adjoint Equations

Equations (10–13) will be integrated using boundary-value techniques. The method used will be to find functions $z_i^1(s)$ and $z_i^2(s)$ that satisfy specified initial conditions. Then a linear combination is formed

$$z_9(s) = z_9^1(s) + \gamma z_9^2(s) \tag{A1}$$

where γ = constant to be determined. The boundary condition $z_9(s_3) = 0$ will be used to give

$$z_9(s_3) = 0 = z_9^1(s_3) + \gamma z_9^2(s_3) \tag{A2}$$

Solving for γ

$$\gamma = -[z_9^1(s_3)/z_9^2(s_3)] \tag{A3}$$

When γ has been determined from Eq. (A3), it is possible to write the general equation.

$$z_i(s) = z_i^1(s) + \gamma z_i^2(s),\ i = 1, \ldots, 12 \tag{A4}$$

The $z_i^1(s)$ and $z_i^2(s)$ may be found by using the following procedure.

1) Integrate the first four equations in Eqs. (10) on $0 \leq s \leq s_1$ to find $z_i^1(s)$, $i = 1,2,3,4$ using the initial conditions

$$z_1^1(0) = 0,\ z_2^1(0) = 0,\ z_3^1(0) = \text{const},\ z_4^1(0) = 0$$

The $z_i^2(s)$, $i = 1,2,3,4$ may be found by repeating the integration of the same equations using the initial conditions

$$z_1^2(0) = 0,\ z_2^2(0) = 0,\ z_3^2(0) = 0,\ z_4^2(0) = 1$$

2) In order to find the $z_i^1(s)$ and $z_i^2(s)$, $i = 5,6,7,8$, it is necessary to perform four separate integrations on the region $s_1 \leq s \leq s_2$ in order to satisfy the condition $z_5(s_2) = 0$.

To determine the $z_i^1(s)$, $i = 5,6,7,8$, it is necessary to find the four $z_i^{11}(s)$ obtained from the integration of the second set of four equations in Eqs. (10) using the initial conditions

$$z_5^{11}(s_1) = 0,\ z_6^{11}(s_1) = z_2^1(s_1),\ z_7^{11}(s_1) = z_3^1(s_1),\ z_8^{11}(s_1) = 0$$

Repeat the integration of the same equations on $s_1 \leq s \leq s_2$ to find a set $z_i^{12}(s)$, $i = 5,6,7,8$, using the initial conditions

$$z_5^{12}(s_1) = 0,\ z_6^{12}(s_1) = 0,\ z_7^{12}(s_1) = 0,\ z_8^{12}(s_1) = 1$$

Now form

$$z_5^1(s) = z_5^{11}(s) + \gamma^* z_5^{12}(s)$$

where γ^* is a constant which is to be determined. Use $z_5(s_2) = 0$ to get

$$0 = z_5^{11}(s_2) + \gamma^* z_5^{12}(s_2)$$

or

$$\gamma^* = -[z_5^{11}(s_2)/z_5^{12}(s_2)]$$

The $z_i^1(s)$, $i = 5,6,7,8$, are now found from the general equation

$$z_i^1(s) = z_i^{11}(s) + \gamma^* z_i^{12}(s),\ i = 5,6,7,8$$

In order to find the $z_i^2(s)$, $i = 5,6,7,8$, it is necessary to find a set of functions $z_i^{21}(s)$ by integrating the second set of four equations in Eqs. (10) on $s_1 \leq s \leq s_2$ using the boundary conditions

$$z_5^{21}(s_1) = 0,\ z_6^{21}(s_1) = z_2^2(s_1),\ z_7^{21}(s_1) = z_3^2(s_1),\ z_8^{21}(s_1) = 0$$

Repeat the integration, finding a set $z_i^{22}(s)$ using the initial condition,

$$z_5^{22}(s_1) = 0,\ z_6^{22}(s_1) = 0,\ z_7^{22}(s_1) = 0,\ z_8^{22}(s_1) = 1$$

Now form $z_5^2(s) = z_5^{21}(s) + \gamma^+ z_5^{22}(s)$. Use the condition $z_5(s_2) = 0$ to find the constant

$$\gamma^+ = -[z_5^{21}(s_2)/z_5^{22}(s_2)]$$

This yields

$$z_i^2(s) = z_i^{21}(s) + \gamma^+ z_i^{22}(s),\ i = 5,6,7,8$$

3) The set of $z_i^1(s)$, $i = 9,10,11,12$ may be found by integrating the third set of four equations in Eqs. (10) using the initial conditions

$$z_9^1(s_2) = -z_1^1(s_1),\ z_{10}^1(s_2) = z_6^1(s_2),\ z_{11}^1(s_2) = z_7^1(s_2)$$

$$z_{12}^1(s_2) = +\omega^2\rho\left[\int_{s_1}^{s_2} u_2(s)ds\right] z_1^1(s_1) - z_4^1(s_1)$$

Also the $z_i^2(s)$, $i = 9,10,11,12$ may be found by integrating the same equations on $s_2 \leq s \leq s_3$ with the initial conditions

$$z_9^2(s_2) = -z_1^2(s_1),\ z_{10}^2(s_2) = z_6^2(s_2),\ z_{11}^2(s_2) = z_7^2(s_2)$$

$$z_{12}^2(s_2) = +\ \omega^2\rho\left[\int_{s_1}^{s_2} u_2(s)ds\right] z_1^2(s_1) - z_4^2(s_1)$$

It is now possible to form the expression

$$z_9(s) = z_9^1(s) + \gamma z_9^2(s)$$

Then

$$\gamma = -[z_9^1(s_3)/z_9^2(s_3)]$$

With γ determined, it is possible to go back and form the expressions

$$z_i(s) = z_i^1(s) + \gamma z_i^2(s),\ i = 1,2,\ldots,12$$

In an analogous manner, the linear boundary-value problem Eqs. (25–26) may be solved. In integrating these equations, however, it is noted that three conditions are given at s_3 and only one condition at 0. Therefore, Eq. (25) is integrated from s_3 backward to 0, just as Eq. (10) was integrated forward from 0.

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